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CONJUGATE SEMI-SYMMETRIC NON-METRIC CONNECTIONS*

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Abstract. In this paper, we study a new semi-symmetric non-metric connection. Firstly, we give its conjugate connection. After the generalized conjugate connection and the semi-conjugate connection of the semi-symmetric non-metric connection are also given. Some properties of the conjugate semi-symmetric non-metric connection are given.

Keywords: Semi-symmetric; non-metric; connection; conjugate connection.

1. Introduction

Let $\widetilde{\nabla}$ be a linear connection in an n-dimensional differentiable manifold M. The torsion tensor $\widetilde{T}$ and the curvature tensor $\widetilde{R}$ of $\widetilde{\nabla}$ are given respectively by

$$\begin{aligned}\widetilde{T} &= \widetilde{\nabla}_X Y - \widetilde{\nabla}_Y X - [X,Y]\\ \widetilde{R} &= \widetilde{\nabla}_X \widetilde{\nabla}_Y Z - \widetilde{\nabla}_Y \widetilde{\nabla}_X Z - \widetilde{\nabla}_{[X,Y]} Z.\end{aligned}$$

The connection $\widetilde{\nabla}$ is symmetric if its torsion tensor $\widetilde{T}$ vanishes, otherwise it is non-symmetric. The connection $\widetilde{\nabla}$ is a metric connection if there is a Riemannian metric g in M such that $\widetilde{\nabla} g = 0$, otherwise it is non-metric. It is well known that a linear connection is symmetric and metric if and only if it is the Levi-Civita connection (cf. M. M. Tripathi [13]). B. G. Schmidt [11] proved that if the holonomy group of $\widetilde{\nabla}$ is a subgroup of the orthogonal group $\mathcal{O}(n)$, then $\widetilde{\nabla}$ is the Levi-Civita connection of a Riemannian metric.

A. Friedmann and J. A. Schouten [7], introduced the concept of semi-symmetric linear connection on a Riemannian manifold. A linear connection $\widetilde{\nabla}$ is said to be a *semi-symmetric connection* if its torsion tensor $\widetilde{T}$ is of the form

$$\widetilde{T}(X,Y) = u(Y)X - u(X)Y, \tag{1.1}$$

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where u is a 1-form associated with the vector field U on M by

$$u(X) = g(X, U). \tag{1.2}$$

H. A. Hayden [8] introduced the concept of a semi-symmetric metric connection on a Riemannian manifold (M, g). A semi-symmetric connection $\widetilde{\nabla}$ is said to be a *semi-symmetric metric connection* if

$$\widetilde{\nabla} g = 0. \tag{1.3}$$

In [14], K. Yano considered semi-symmetric metric connection and studied some of its properties. He proved that a Riemannian manifold admitting a semi-symmetric metric connection has vanishing curvature tensor if it is conformally flat. He also proved that a Riemannian manifold is of constant curvature if and only if it admits a semi-symmetric metric connection for which the manifold is a group manifold, where a group manifold is a differentiable manifold admitting a linear connection $\widetilde{\nabla}$ such that its curvature tensor $\widetilde{R}$ vanishes and its torsion tensor $\widetilde{T}$ is covariantly constant with respect to $\widetilde{\nabla}$ (cf. M. M. Tripathi [13]).

N. S. Agashe and M. R. Chafle [1], introduced the notion of a semi-symmetric non-metric connection on a Riemannian manifold (M, g). A semi-symmetric connection $\widetilde{\nabla}$ is said to be a *semi-symmetric non-metric connection* if

$$\widetilde{\nabla} g \neq 0. \tag{1.4}$$

In [2], N. S. Agashe and M. R. Chafle studies some of its properties and submanifolds of a Riemannian manifold with semi-symmetric non-metric connections. J. Sengupta, U. C. De and T. Q. Binh in [12], generalizes the semi-symmetric non-metric connection introduced in [1]. They determine the relationships between the curvature tensor $\widetilde{R}$ of M, with respect to the semi-symmetric non-metric connection $\widetilde{\nabla}$ and the curvature tensor field R, with respect to the Riemannian connection. Further, the first and second Bianchi identities associated with the semi-symmetric non-metric connection are obtained. Some properties of the Weyl projective curvature tensor with respect to the semi-symmetric non-metric connection $\widetilde{\nabla}$ are also studied.

Geometry of conjugate connections is a natural generalization of geometry of Levi-Civita connections from Riemannian manifolds theory. Since conjugate connections arise from affine differential geometry and from geometric theory of statistical inferences, many studies have been carried out in the recent years [3]. In this paper, firstly we introduce a new semi-symmetric non-metric. After, we establish conjugate connection, generalized conjugate connection and semi-conjugate of the new semi-symmetric non-metric connection.

The outline of the paper is as follows. In section 2., we recall some basis notions about conjugate connections. In section 3., we prove the existence of a new semi-symmetric non-metric connection which generalize the semi-symmetric non-metric connections given by Agashe and Chafle [1], and Sengupta *and al.* [12].

In section 4., we establish the expression of the conjugate connection of the new semi-symmetric non-metric connection. In section 5. and section 6., the generalized conjugate connection and the semi-conjugate connection of the new semi-symmetric non-metric connection are given, respectively. Note that the generalized conjugate connection and the semi-conjugate conjugate are both generalizations of the conjugate connection.

2. Preliminaries

We assume that all the objects are smooth throughout this paper. In this section, we recall the basic notions of conjugate connections.

Let (M, g) be a Riemannian manifold and ∇ an affine connection on M. A connection ∇^* is called *conjugate connection* of ∇ with respect to the metric g if

$$g(\nabla^*_X Y, Z) = X \cdot g(Y, Z) - g(Y, \nabla_X Z) \tag{2.1}$$

for arbitrary $X, Y, Z \in \Gamma(TM)$. The triple (g, ∇, ∇^*) satisfying (2.1) is called *dualistic structure* on M. It is easy to see that $(\nabla^*)^* = \nabla$ and that $\frac{1}{2}(\nabla + \nabla^*)$ is the Levi-Civita on (M, g). A geometric interpretation of the equation (2.1) can be found in ([10], Proposition 4.5).

Proposition 2.1. *[6] The torsion tensors T and T^* of ∇ and ∇^*, respectively, satisfy:*

$$g(T(X, Y), Z) = g(T^*(X, Y), Z) + (\nabla^* g)(X, Y, Z) - (\nabla^* g)(Y, X, Z)$$

for any $X, Y, Z \in \Gamma(TM)$.

Corollary 2.1. *If $\nabla^* g = 0$, then $T = T^*$.*

Let $C(X, Y, Z) = \nabla_X g(Y, Z)$ the cubic form of (∇, g) and $C^*(X, Y, Z) = \nabla^*_X g(Y, Z)$ the cubic form of (∇^*, g). We have the following property:

Proposition 2.2. *[6] The cubic form of (∇, g) is symmetric if and only if the cubic form of (∇^*, g) is symmetric.*

Corollary 2.2. *If $\nabla^* g$ is symmetric and ∇^* is torsion free, then ∇g is symmetric and ∇ is torsion free too.*

The notion of conjugate connection has an origin in affine differential geometry. If (∇, g) is the pair of induced affine connection and the affine metric on a nondegenerate hypersurface M in the affine space $\mathbb{R}^{n+1}$, the conjugate connection ∇^* is the affine connection induced on M by an immersion of M into the dual affine space, called the conormal immersion [10]. It is worth noting that this notion does play an important role also in mathematical statistics (see [3]). It was introduced into the subject, apparently without any regard to what had been known in classical affine differential geometry, a fact that makes the relationship more intriguing. For other studies of Riemannian manifolds with conjugate connections (see also [5, 6] and references therein).

3. Semi-symmetric non-metric connection

In this subsection, we prove the following theorem.

Theorem 3.1. *Let M be an m-dimensional Riemannian manifold equipped with the Levi-Civita connection $\mathring{\nabla}$ of its Riemannian metric g. Let f be a function on M and u, u_1 are 1-forms associated with the vector fields U, U_1 on M defined by*

$$u(X) = g(U, X), \quad u_1(X) = g(U_1, X). \tag{3.1}$$

Then there exists a unique connection $\widetilde{\nabla}$ in M given by

$$\widetilde{\nabla}_X Y = \mathring{\nabla}_X Y + u(Y)X - g(X, Y)U - fg(X, Y)U_1 \tag{3.2}$$

which satisfies

$$\widetilde{T}(X, Y) = u(Y)X - u(X)Y \tag{3.3}$$

and

$$\Big(\widetilde{\nabla}_X g\Big)(Y, Z) = f\Big(u_1(Y)g(X, Z) + u_1(Z)g(X, Y)\Big) \tag{3.4}$$

where $\widetilde{T}$ is the torsion tensor of $\widetilde{\nabla}$.

Proof. Let $\widetilde{\nabla}$ be a linear connection defined on M by

$$\widetilde{\nabla}_X Y = \mathring{\nabla}_X Y + \theta(X, Y) \tag{3.5}$$

where $\mathring{\nabla}$ is the Levi-Civita connection and θ is a tensor of type (1,2) defined on M such that $\widetilde{\nabla}$ satisfies (3.3) and (3.4).
From (3.5), we have

$$\widetilde{T}(X, Y) = \theta(X, Y) - \theta(Y, X). \tag{3.6}$$

Denote

$$G(X, Y, Z) = \Big(\widetilde{\nabla}_X g\Big)(Y, Z) \tag{3.7}$$

From (3.5) and (3.7), we have:

$$\begin{aligned} \widetilde{\nabla}_X \Big(g(Y, Z)\Big) &= \Big(\widetilde{\nabla}_X g\Big)(Y, Z) + g(\widetilde{\nabla}_X Y, Z) + g(Y, \widetilde{\nabla}_X Z) \\ &= G(X, Y, Z) + g(\mathring{\nabla}_X Y, Z) + g(\theta(X, Y), Z) \\ &\quad + g(Y, \mathring{\nabla}_X Z) + g(Y, \theta(X, Z)) \end{aligned}$$

which implies that

$$g(\theta(X, Y), Z) + g(Y, \theta(X, Z)) = -G(X, Y, Z). \tag{3.8}$$

From (3.6) and (3.8), we have:

$$\begin{aligned}
& g(\widetilde{T}(X,Y),Z)+g(\widetilde{T}(Z,X),Y)+g(\widetilde{T}(Z,Y),X) \\
= \;& g(\theta(X,Y),Z)-g(\theta(Y,X),Z)+g(\theta(Z,X),Y) \\
& -g(\theta(X,Z),Y)+g(\theta(Z,Y),X)-g(\theta(Y,Z),X) \\
= \;& g(\theta(X,Y),Z)-g(\theta(X,Z),Y)+G(Y,X,Z)-G(Z,X,Y) \\
= \;& 2g(\theta(X,Y),Z)+G(X,Y,G)+G(Y,X,Z)-G(Z,X,Y).
\end{aligned}$$

From (3.7), we have:

$$\begin{aligned}
& g(\widetilde{T}(X,Y),Z)+g(\widetilde{T}(Z,X),Y)+g(\widetilde{T}(Z,Y),X) \\
= \;& 2g(\theta(X,Y),Z)+\Big(\widetilde{\nabla}_X g\Big)(Y,Z)+\Big(\widetilde{\nabla}_Y g\Big)(X,Z)-\Big(\widetilde{\nabla}_Z g\Big)(X,Y).
\end{aligned}$$

From (3.4), we have:

$$\begin{aligned}
& g(\widetilde{T}(X,Y),Z)+g(\widetilde{T}(Z,X),Y)+g(\widetilde{T}(Z,Y),X)=2g(\theta(X,Y),Z) \\
& +f\Big(u_1(Y)g(X,Z)+u_1(Z)(X,Y)\Big) \\
& +f\Big(u_1(X)g(Y,Z)+u_1(Z)(X,Y)\Big) \\
& -f\Big(u_1(X)g(Y,Z)+u_1(Y)(X,Z)\Big)
\end{aligned}$$

which implies that

$$\theta(X,Y)=\frac{1}{2}\Big(\widetilde{T}(X,Y)+\widetilde{T}'(X,Y)+\widetilde{T}'(Y,X)\Big)-fg(X,Y)U_1 \tag{3.9}$$

where

$$g(\widetilde{T}'(X,Y),Z)=g(\widetilde{T}(Z,X),Y). \tag{3.10}$$

Using (3.1) and (3.3) in (3.10), we get

$$g(\widetilde{T}'(X,Y),Z)=u(X)g(Y,Z)-g(X,Y)g(U,Z).$$

which implies that

$$\widetilde{T}'(X,Y)=u(X)Y-g(X,Y)U. \tag{3.11}$$

Using (3.3) and (3.11) in (3.9), we get:

$$\theta(X,Y)=u(Y)X-g(X,Y)U-fg(X,Y)U_1. \tag{3.12}$$

Substituting (3.12) in (3.5), we have

$$\widetilde{\nabla}_X Y=\mathring{\nabla}_X Y+u(Y)X-g(X,Y)U-fg(X,Y)U_1.$$

Conversely, a linear connection given by (3.2) satisfies the condition (3.3) and (3.4). $\square$

The semi-symmetric non-metric connection in Theorem (3.1) is a generalization of the semi-symmetric non-metric given by N. S. Agashe and M. R. Chafle [1] (for $f = -1$ and $U = U_1$) and the semi-symmetric non-metric given by J. Sengupta and al., [12] (for $f = -1$). When the function f is equal to zero, then we obtain a semi-symmetric metric connection.

4. Conjugate semi-symmetric non-metric connection

In the following proposition, we obtain the conjugate connection of the semi-symmetric non-metric connection given in Theorem 3.1

Proposition 4.1. *The conjugate connection of the semi-symmetric non-metric connection defined by*

$$\widetilde{\nabla}_X Y = \mathring{\nabla}_X Y + u(Y)X - g(X,Y)U - fg(X,Y)U_1$$

is the semi-symmetric non-metric connection given by

$$\widetilde{\nabla}^*_X Y = \mathring{\nabla}_X Y + u(Y)X - g(X,Y)U + fu_1(Y)X \tag{4.1}$$

where $u(Y) = g(Y,U)$ *and* $u_1(Y) = g(U,U_1)$.

Proof. Using (2.1) and (3.2) we have

$$\begin{aligned} g(Z,\widetilde{\nabla}^*_X Y) &= X\cdot g(Z,Y) - g(\widetilde{\nabla}_X Z, Y) \\ &= g(\mathring{\nabla}_X Z, Y) + g(Z,\mathring{\nabla}_X Y) - g(\mathring{\nabla}_X Z, Y) \\ &\quad -u(Z)g(X,Y) + g(X,Z)g(U,Y) + fg(X,Z)g(U_1,Y) \\ &= g(Z,\mathring{\nabla}_X Y) - g(Z,U)g(X,Y) \\ &\quad +g(Z,X)u(Y) + fg(Z,X)u_1(Y). \end{aligned}$$

Hence

$$\widetilde{\nabla}^*_X Y = \mathring{\nabla}_X Y + u(Y)X - g(X,Y)U + fu_1(Y)X.$$

□

We call this new connection $\widetilde{\nabla}^*$ the conjugate semi-symmetric non-metric connection of $\widetilde{\nabla}$ with respect to g.

Remark 4.1. Let $\widetilde{\nabla}^*$ the conjugate semi-symmetric non-metric connection of the semi-symmetric connection $\widetilde{\nabla}$ with respect to g. Then

$$X\cdot g(Y,Z) = g(\mathring{\nabla}_X Y, Z) + g(Y, \widetilde{\nabla}^*_X Z).$$

Corollary 4.1. *In (3.2), if* $f = -1$*, we obtain the semi-symmetric non-metric connection* $\widetilde{\nabla}$ *given by J. Sengupta and al. [12] as*

$$\widetilde{\nabla}_X Y = \mathring{\nabla}_X Y + u(Y)X - g(X,Y)U + g(X,Y)U_1. \tag{4.2}$$

Then its conjugate connection is also a semi-symmetric non-metric connection given by

$$\widetilde{\nabla}^*_X Y = \mathring{\nabla}_X Y + \omega(Y)X - g(X,Y)U \tag{4.3}$$

where $\omega = u - u_1$ is a 1-form in M.

Corollary 4.2. *In (3.2), if $f = -1$ and $U = U_1$, we obtain the semi-symmetric non-metric connection $\widetilde{\nabla}$ given by N. S. Agashe and M. R. Chafle [1] as*

$$\widetilde{\nabla}_X Y = \mathring{\nabla}_X Y + u(Y)X. \tag{4.4}$$

Then its conjugate connection is also a semi-symmetric non-metric connection given by

$$\widetilde{\nabla}^*_X Y = \mathring{\nabla}_X Y - g(X,Y)U \tag{4.5}$$

We also have also the following observation.

Proposition 4.2. *Let $\widetilde{\nabla}$ be the semi-symmetric non-metric connection (3.2) and $\widetilde{\nabla}^*$ its conjugate connection (4.1). Then we have*

$$\widetilde{\nabla}^*_X Y - \widetilde{\nabla}_X Y = f\Big(u_1(Y)X + g(X,Y)U_1\Big) \tag{4.6}$$

and

$$\widetilde{T}^*(X,Y) - \widetilde{T}(X,Y) = f\Big(u_1(Y)X - u_1(X)Y\Big) \tag{4.7}$$

where $\widetilde{T}^*$ and $\widetilde{T}$ are the torsion tensors of $\widetilde{\nabla}^*$ and $\widetilde{\nabla}$, respectively.

Proof. The relation (4.6) is obtain by using (3.2) and (4.1). The relation (4.7) is obtain by a straightforward calculation from the following definition

$$\widetilde{T}^* = \widetilde{\nabla}^*_X Y - \widetilde{\nabla}^*_Y X - [X,Y].$$

□

Corollary 4.3. *The semi-symmetric non metric connection (4.1) satisfies*

$$(\widetilde{\nabla}^*_X g)(Y,Z) = -f\Big(u_1(Y)g(X,Z) + u_1(Z)g(X,Y)\Big).$$

When $f = 0$, we obtain a semi-symmetric metric connection. The relation between the semi-symmetric metric connection $\widetilde{\nabla}$ and the Levi-Civita connection $\mathring{\nabla}$ of (M,g) is given by K. Yano [14] as

$$\widetilde{\nabla}_X Y = \mathring{\nabla}_X Y + u(Y)X - g(X,Y)U \tag{4.8}$$

where

$$u(X) = g(X,U)$$

and X, Y, U are vector fields on M. Therefore, we have the following.

Proposition 4.3. *If $\widetilde{\nabla}$ is a semi-symmetric metric connection on M, so is its dual connection.*

Proof. Using (2.1) and (4.8), we have

$$\begin{aligned} g(Z,\widetilde{\nabla}^*_X Y) &= X\cdot g(Z,Y) - g(\widetilde{\nabla}_X Z,Y) \\ &= g(\mathring{\nabla}_X Z,Y) + g(Z,\mathring{\nabla}_X Y) - g(\widetilde{\nabla}_X Z,Y) \\ &= g(\mathring{\nabla}_X Z,Y) + g(Z,\mathring{\nabla}_X Y) \\ &\quad -g(\mathring{\nabla}_X Z,Y) - g(u(Z)X,Y) + g(X,Z)g(U,Y) \\ &= g(Z,\mathring{\nabla}_X Y) - u(Z)g(X,Y) + g(U,Y)g(X,Z) \\ &= g(Z,\mathring{\nabla}_X Y) - g(X,Y)g(Z,U) + g(Y,U)g(X,Z). \end{aligned}$$

Hence, we obtain:

$$\widetilde{\nabla}^*_X Y = \mathring{\nabla}_X Y + u(Y)X - g(X,Y)U.$$

□

5. Generalized conjugate connection of a semi-symmetric non-metric connection

Let (M,g) be a Riemannian manifold of dimension $n(\geq 2)$. Let $\mathcal{C}$ be a conformal structure on M determined by g and ∇ a torsion-free affine connection on M. We call the pair $(\nabla,\mathcal{C})$ a *Weyl structure* on M if there exists a 1-form u such that

$$(\nabla_X g)(Y,Z) = -u(X)g(Y,Z).$$

We call the triplet (M,∇,g) a *Weyl manifold* and the connection ∇ a *Weyl connection*. It is easy to show that u is determined uniquely since $m(\geq 2)$. A weyl structure or a Weyl manifold is said to be *locally trivial* if the 1-form u is closed [9].

Definition 5.1. [4] Let (M,g) be a Riemannian manifold, ∇ an affine connection on M, and u a 1-form on M. The *generalized conjugate connection* $\overline{\nabla}^*$ of ∇ with respect to g by u is defined by

$$g(Z,\overline{\nabla}^*_X Y) = X\cdot g(Z,Y) - g(\nabla_X Z,Y) + u(X)g(Z,Y). \tag{5.1}$$

The generalized conjugate connection is a generalization of conjugate connection introduced in Weyl geometry to characterize Weyl connections.

Proposition 5.1. *Let (M,g) be a Riemannian manifold and $\widetilde{\nabla}$ the semi-symmetric non-metric connection on M defined by*

$$\widetilde{\nabla}_X Y = \mathring{\nabla}_X Y + u(Y)X - g(X,Y)U - fg(X,Y)U_1.$$

Then its generalized conjugate connection $\overline{(\widetilde{\nabla})}^*$ is given by

$$\overline{(\widetilde{\nabla})}^*_X Y = \mathring{\nabla}_X Y + u(X)Y + u(Y)X - g(X,Y)U + fu_1(Y)X. \tag{5.2}$$

Proof. From equations (3.2) and (5.1), we obtain

$$\begin{aligned} g(Z, \overline{(\tilde{\nabla})}^*_X Y) &= g(Z, \overset{\circ}{\nabla}_X Y) + u(X)g(Z,Y) + u(Y)g(Z,X) \\ &\quad - g(X,Y)g(Z,U) + fu_1(Y)g(Z,X). \end{aligned}$$

Hence, we have equation (5.2). □

Proposition 5.2. *Let (M, g) be a Riemannian manifold, $\tilde{\nabla}$ a semi-symmetric non-metric connection on M and $\tilde{\nabla}^*$ the conjugate connection of $\tilde{\nabla}$ with respect to g. Suppose that an affine connection ∇' is projectively equivalent to $\tilde{\nabla}$ by u. Then the generalized conjugate connection $\overline{\nabla'}^*$ of ∇' by u is dual-projectively equivalent to $\tilde{\nabla}^*$ by u with respect to g.*

Proof. From (2.1) we have

$$X \cdot g(Y,Z) = g(\tilde{\nabla}_X Y, Z) + g(Y, \tilde{\nabla}^*_X Z). \tag{5.3}$$

From (5.1), we have

$$X \cdot g(Y,Z) = g(\nabla'_X Y, Z) + g(Y, \overline{\nabla'}^*_X Z) - u(X)g(Y,Z). \tag{5.4}$$

Recall that affine connections ∇' and $\tilde{\nabla}$ are said to be projectively equivalent if there exists a 1-form u on M such that

$$\nabla'_X Y = \tilde{\nabla}_X Y + u(Y)X + u(X)Y. \tag{5.5}$$

From Equations (5.4) and (5.5), we obtain

$$X \cdot g(Y,Z) = g(\tilde{\nabla}_X Y, Z) + g(Y, \overline{\nabla'}^*_X Z) + u(Y)g(X,Z). \tag{5.6}$$

From Equations (5.3) and (5.6), we obtain

$$g(Y, \tilde{\nabla}^*_X Z) = g(Y, \overline{\nabla'}^*_X Z) + u(Y)g(X,Z).$$

Hence

$$\overline{\nabla'}^*_X Z = \tilde{\nabla}^*_X Z - g(X,Z)U.$$

This implies that $\overline{\nabla'}^*$ is dual-projectively equivalent to $\tilde{\nabla}^*$ by u with respect to g. □

Following similar steps as in above Proposition, we obtain the following proposition.

Proposition 5.3. *Let (M, g) be a Riemannian manifold, $\tilde{\nabla}$ a semi-symmetric non-metric connection on M and $\tilde{\nabla}^*$ the conjugate connection of $\tilde{\nabla}$ with respect to g. Suppose that an affine connection ∇' is dual-projectively equivalent to $\tilde{\nabla}$ by u. Then the generalized conjugate connection $\overline{\nabla'}^*$ of ∇' by u is projectively equivalent to $\tilde{\nabla}^*$ by u with respect to g.*

Proof. From (2.1) we have

$$X \cdot g(Y,Z) = g(\tilde{\nabla}_X Y, Z) + g(Y, \widetilde{\nabla}^*_X Z). \tag{5.7}$$

From (5.1), we have

$$X \cdot g(Y,Z) = g(\nabla'_X Y, Z) + g(Y, \overline{\nabla'}^*_X Z) - u(X)g(Y,Z). \tag{5.8}$$

Recall that affine connections ∇' and $\widetilde{\nabla}$ are said to be dual-projectively equivalent if there exists a 1-form u on M such that

$$\nabla'_X Y = \widetilde{\nabla}_X Y - g(X,Y)U. \tag{5.9}$$

From Equations (5.8) and (5.9), we obtain

$$\begin{aligned} X \cdot g(Y,Z) &= g(\widetilde{\nabla}_X Y, Z) + g(Y, \overline{\nabla'}^*_X Z) \\ &\quad -u(X)g(Y,Z) - g(U,Z)g(X,Y). \end{aligned} \tag{5.10}$$

From Equations (5.7) and (5.10), we obtain

$$g(Y, \widetilde{\nabla}^*_X Z) = g(Y, \overline{\nabla'}^*_X Z) - u(X)g(Y,Z) - u(Z)g(X,Y)$$

Hence

$$\overline{\nabla'}^*_X Z = \widetilde{\nabla}^*_X Z + u(Z)X + u(X)Z.$$

This implies that $\overline{\nabla'}^*$ is projectively equivalent to $\widetilde{\nabla}^*$ with respect to u. □

Let (M,g) be a Riemannian manifold, ∇ an affine connection on M, and ϕ a smooth function on M. We consider a conformal change of the metric $\bar{g} := e^{\phi} g$. Denote by $\overline{\nabla}^*$ the conjugate connection of ∇ with respect to the conformal metric $\bar{g}$, i.e.,

$$X \cdot \bar{g}(Y,Z) = \bar{g}(\nabla_X Y, Z) + \bar{g}(Y, \overline{\nabla}^*_X Z).$$

Then we have:

$$X \cdot g(Y,Z) = gg(\nabla_X Y, Z) + g(Y, \overline{\nabla}^*_X Z) - d\phi(X)g(Y,Z).$$

This implies that $\overline{\nabla}^*$ is the generalized conjugate connection of ∇ with respect to g by $d\phi$. Let u be a 1-form on M. Set

$$(\bar{g}, \bar{u}) := (e^{\phi} g, u - d\phi).$$

The pair $(\bar{g}, \bar{u})$ is called a *gauge transformation* of (g,u).

Proposition 5.4. *[4] The generalized conjugate connection is invariant under gauge transformation. That is, $\overline{\nabla}^*$ is the generalized conjugate connection of ∇ with respect to g by u if and only if $\overline{\nabla}^*$ is the conjugate connection of ∇ with respect to $\bar{g}$.*

The notion of Weyl manifolds was proposed by Weyl (see [9] and references therein) to construct a generalized structure of space-time geometry which would unify the laws of gravitation and of electromagnetism. His approach did not succeed in physics, but Weyl manifolds have been studied in mathematics (see [9] and references therein).

6. Semi-Weyl geometry

Recall that the triplet (M, g, ∇) is *statistical manifold* if ∇ is a torsion free affine connection and ∇g is symmetric, that is, if the following equation holds:

$$\Big(\nabla_X g\Big)(Y, Z) = \Big(\nabla_Y g\Big)(X, Y).$$

The affine connection ∇ is said to be *compatible* with g. If the connection ∇ is the Levi-Civita connection of (M, g), then $\nabla g = 0$. If ∇^* is another affine connection on M such that ∇ and ∇^* are conjugate, then ∇^* is also torsion free and $\nabla^* g$ is symmetric. In this case (M, g, ∇^*) is also called the *dual statistical manifold* of (M, g, ∇) and we say that (g, ∇, ∇^*) is a dualistic structure on M.

H. Matsuzoe [9] proposes a common generalization of statistical manifolds and Weyl manifolds: both are Riemannian manifolds equipped with a torsion-free connection which does not preserve the metric, but satisfies a weaker compatibility condition. For this, Matsuzoe [9] introduce the notion of semi-Weyl manifolds.

Definition 6.1. Let (M, g) be a Riemannian manifold of dimension $n(\geq 2)$ and ∇ a torsion-free affine connection on M. Denote by $\mathcal{C}$ the conformal structure on M defined by g. We call the pair $(\nabla, \mathcal{C})$ a *semi-Weyl structure* on M if there exists a 1-form u such that

$$(\nabla_X g)(Y, Z) + u(X)g(Y, Z) = (\nabla_Y g)(X, Z) + u(Y)g(X, Z).$$

We call the triplet (M, ∇, g) a *semi-Weyl manifold* and the affine connection ∇ is called semi-Weyl compatible with g by u.

If $u = 0$ then (M, ∇, g) is a statistical manifold. We introduce a $(0, 3)$- tensor C by

$$C(X, Y, Z) := (\nabla_X g)(Y, Z) + u(X)g(Y, Z).$$

When C vanishes everywhere, then (M, ∇, g) is a Weyl manifold. On a semi-Weyl manifold (M, ∇, g), another affine connection $\widehat{\nabla}^*$ can be defined by the following.

Definition 6.2. [4] Let (M, ∇, g) be a semi-Weyl manifold. The *semi-conjugate connection* (or *semi-dual connection*) $\widehat{\nabla}^*$ of ∇ with respect to g by u is defined as

$$g(Z, \widehat{\nabla}^*_X Y) = X \cdot g(Z, Y) - g(\nabla_X Z, Y) - u(Y)g(X, Z). \tag{6.1}$$

The semi-conjugate connection is another generalization of conjugate connection. Matsuzoe showed the following proposition in [9].

Proposition 6.1. *Let (M, ∇, g) be a semi-Weyl manifold and $\widehat{\nabla}^*$ the semi-conjugate connection of ∇. Then $(M, \widehat{\nabla}^*, g)$ is a statistical manifold.*

Next, we calculate the semi-conjugate connection associated with the semi-symmetric non-metric connection given in Theorem 3.1

Proposition 6.2. *Let (M,g) be a Riemannian manifold and $\tilde{\tilde{\nabla}}$ the semi-symmetric non-metric connection on M defined by*

$$\tilde{\tilde{\nabla}}_X Y = \mathring{\nabla}_X Y + u(Y)X - g(X,Y)U - fg(X,Y)U_1.$$

Then its semi-conjugate connection $\widehat{(\tilde{\tilde{\nabla}})}^*$ is given by

$$\widehat{(\tilde{\tilde{\nabla}})}^*_X Y = \mathring{\nabla}_X Y + u(Y)X - g(X,Y)U + fu_1(Y)X. \tag{6.2}$$

Proof. From equations (3.2) and (6.1), we obtain

$$g(Z, \widehat{(\tilde{\tilde{\nabla}})}^*_X Y) \;=\; g(Z, \mathring{\nabla}_X Y) - g(X,Y)g(Z,U) + fg(U_1,Y)g(Z,X).$$

Hence, we have equation (6.2) □

Proposition 6.3. *Let (M,g) be a Riemannian manifold, $\tilde{\tilde{\nabla}}$ a semi-symmetric non-metric connection on M and $\tilde{\tilde{\nabla}}^*$ the conjugate connection of $\tilde{\tilde{\nabla}}$ with respect to g. Suppose that an affine connection ∇' is projectively equivalent to $\tilde{\tilde{\nabla}}$ by u. Then the semi-conjugate connection $\widehat{\nabla'}^*$ of ∇' by u is given by*

$$\widehat{\nabla'}^*_X Y = \tilde{\tilde{\nabla}}^*_X Y - u(Y)X - u(X)Z - g(X,Y)U. \tag{6.3}$$

Proof. From (2.1) we have

$$X \cdot g(Y,Z) = g(\tilde{\tilde{\nabla}}_X Y, Z) + g(Y, \tilde{\tilde{\nabla}}^*_X Z). \tag{6.4}$$

From (6.1), we have

$$X \cdot g(Y,Z) = g(\nabla'_X Y, Z) + g(Y, \widehat{\nabla'}^*_X Z) + u(Z)g(X,Y). \tag{6.5}$$

Recall that affine connections ∇' and $\tilde{\tilde{\nabla}}$ are said to be projectively equivalent if there exists a 1-form u on M such that

$$\nabla'_X Y = \tilde{\tilde{\nabla}}_X Y + u(Y)X + u(X)Y. \tag{6.6}$$

From Equations (6.5) and (6.6), we obtain

$$\begin{aligned} X \cdot g(Y,Z) \;=\; & g(\tilde{\tilde{\nabla}}_X Y, Z) + g(Y, \widehat{\nabla'}^*_X Z) + u(Z)g(X,Y) \\ & + u(Y)g(X,Z) + u(X)g(Y,Z). \end{aligned} \tag{6.7}$$

From Equations (6.4) and (6.7), we obtain

$$g(Y, \tilde{\tilde{\nabla}}^*_X Z) = g(Y, \widehat{\nabla'}^*_X Z) + u(Z)g(X,Y) + u(Y)g(X,Z) + u(X)g(Y,Z).$$

Hence

$$\widehat{\nabla'}^*_X Z = \tilde{\tilde{\nabla}}^*_X Z - u(Z)X - u(X)Y - g(X,Z)U.$$

□

Proposition 6.4. *Let (M, g) be a Riemannian manifold, $\widetilde{\nabla}$ a semi-symmetric non-metric connection on M and $\widetilde{\nabla}^*$ the conjugate connection of $\widetilde{\nabla}$ with respect to g. Suppose that an affine connection ∇' is dual-projectively equivalent to $\widetilde{\nabla}$ by u. Then the semi-conjugate connection $\widehat{\nabla'}^*$ of ∇' by u coincides with $\widetilde{\nabla}^*$ on M.*

Proof. From (2.1) we have

$$X \cdot g(Y, Z) = g(\widetilde{\nabla}_X Y, Z) + g(Y, \widetilde{\nabla}^*_X Z). \tag{6.8}$$

From (6.1), we have

$$X \cdot g(Y, Z) = g(\nabla'_X Y, Z) + g(Y, \widehat{\nabla'}^*_X Z) + u(Z)g(X, Y). \tag{6.9}$$

Recall that affine connections ∇' and $\widetilde{\nabla}$ are said to be dual-projectively equivalent if there exists a 1-form u on M such that

$$\nabla'_X Y = \widetilde{\nabla}_X Y - g(X, Y)U. \tag{6.10}$$

From Equations (6.9) and (6.10), we obtain

$$X \cdot g(Y, Z) = g(\widetilde{\nabla}_X Y, Z) + g(Y, \widehat{\nabla'}^*_X Z) + u(Z)g(X, Y) - g(X, Y)g(U, Z). \tag{6.11}$$

From Equations (6.9) and (6.11), we obtain

$$g(Y, \widetilde{\nabla}^*_X Z) = g(Y, \widehat{\nabla'}^*_X Z).$$

Hence

$$\widehat{\nabla'}^*_X Z = \widetilde{\nabla}^*_X Z.$$

□

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